Istituzioni di Matematiche
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<u>Teorema</u> (I° sostituzione)

Sia $g : (a,b) \longrightarrow \mathbb{R}$ dotata di primitive

e $f : (c,d) \longrightarrow (a,b)$ derivabile

$\Longrightarrow$ $g(f(x)) \cdot f'(x)$ ammette primitive, e vale

$$\int g(f(x)) \cdot f'(x)\, dx = \int g(t)\, dt \Big]_{t=f(x)}$$

<u>Dim</u> Usiamo il principio della doppia inclusione.

Prima proviamo $\int g(t)\, dt \Big]_{t=f(x)} \subseteq \int g(f(x)) f'(x)\, dx$

Infatti, sia $H \in \int g(t)\, dt \Big]_{t=f(x)}$

$\Rightarrow$ $(H(t))' = (H(f(x)))' =$ (derivaz. funz. comp)

$$= H'(f(x)) \cdot f'(x) = (H \text{ è primitiva di } g)$$

$$= g(f(x)) \cdot f'(x)$$

$$\Longrightarrow \quad H \in \int g(f(x)) \cdot f'(x)\, dx$$

Per l'inclusione inversa, considera

$$H \in \int g(f(x)) \cdot f'(x)\, dx$$

Ma dalla prima inclusione, sappiamo già che

ogni $\quad G \in \int g(t)\, dt \Big]_{t=f(x)}$ è anche primitiva di

$$g(f(x))\, f'(x)$$

$$\Longrightarrow \exists\, c \in \mathbb{R} : \quad H(x) = G(t)\Big]_{t=f(x)} + c \in \int g(t)\, dt \Big]_{t=f(x)}$$

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<u>Applicazioni</u> :

(1) $\quad \forall n \in \mathbb{N}$

$$\int tg^n x\, dx = \int tg^{n-2} x \cdot tg^2 x\, dx =$$

$$= \int tg^{n-2} x \cdot \left(\frac{1}{?} - 1 \right) dx =$$

$$= \int tg^{n-2}x \cdot \left(\frac{1}{\cos^2 x} - 1\right) dx =$$

$$= \int \underbrace{tg^{n-2}x}_{(tg\,x)^{n-2}} \cdot \underbrace{\frac{1}{\cos^2 x}}_{(tg\,x)'} dx - \int tg^{n-2}x\, dx$$

$$\underset{1° \text{ sost.}}{=} \left.\int t^{n-2}\, dt\right]_{t=tg\,x} - \int tg^{n-2}x\, dx$$

$$= \frac{t^{(n-2)+1}}{(n-2)+1} - \int tg^{n-2}x\, dx$$

$$= \frac{tg^{n-1}x}{n-1} - \int tg^{n-2}x\, dx$$

Conclusione :

$$\boxed{\int tg^{n}x\, dx = \frac{tg^{n-1}x}{n-1} - \int tg^{n-2}x\, dx}$$

caso particolare

$(n=2)$ $$\int tg^2 x\, dx = tg\,x - \int 1\, dx = tg\,x - x + c$$

Integrali Fratti (Importante)

Vogliamo determinare $\int \frac{P(x)}{Q(x)}\,dx$ dove $P(x)$ e $Q(x)$ sono due generici polinomi

Passo 1 (che servirà successivamente) Utilizzo integrazione per parti

$\forall n \in \mathbb{N}$

$$\int \frac{1}{(1+x^2)^{n-1}}\,dx = \int \underbrace{1}_{f'(x)} \cdot \underbrace{\frac{1}{(1+x^2)^{n-1}}}_{g(x)}\,dx =$$

$$\overset{(IP)}{=} x \cdot \frac{1}{(1+x^2)^{n-1}} - \int x \cdot (1-n)\,\frac{2x}{(1+x^2)^n}\,dx$$

$$= \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int \frac{(1+x^2)-1}{(1+x^2)^n} =$$

$$= \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int \frac{\cancel{1+x^2}}{(1+x^2)^{\cancel{n}\,(n-1)}}\,dx - 2(n-1)\int \frac{1}{(1+x^2)^n}$$

Ossia, abbiamo ottenuto

$$\int \frac{1}{(1+x^2)^{n-1}}\,dx = \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int \frac{1}{(1+x^2)^{n-2}}\,dx - 2(n-1)\int \frac{1}{(1+x^2)^n}\,dx$$

$$\underbrace{\int \frac{1}{(1+x^2)^{n-1}}\,dx}_{\text{porto al 2° membro}} = \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int \frac{1}{(1+x^2)^{n-1}}\,dx - \underbrace{2(n-1)\int \frac{1}{(1+x^2)^{n}}\,dx}_{\text{porto al 1° membro}}.$$

Si ha

$$2(n-1)\int \frac{1}{(1+x^2)^{n}}\,dx = \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\underbrace{\int \frac{1}{(1+x^2)^{n-1}}\,dx} - \underbrace{\int \frac{1}{(1+x^2)^{n-1}}\,dx}.$$

Sono uguali

$$= \frac{x}{(1+x^2)^{n-1}} + \left[2(n-1) - 1\right]\int \frac{1}{(1+x^2)^{n-1}}\,dx$$

$$= \frac{x}{(1+x^2)^{n-1}} + (2n-3)\int \frac{1}{(1+x^2)^{n-1}}\,dx$$

Abbiamo ottenuto

$$\boxed{\int \frac{1}{(1+x^2)^{n}}\,dx = \frac{1}{2(n-1)}\,\frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{1}{(1+x^2)^{n-1}}\,dx}$$

Passo 1 Calcoliamo

$$\int \frac{px+q}{ax^2+bx+c}\,dx$$

Sia $\Delta = b^2 - 4ac$

Caso 1 $(\Delta > 0) \implies ax^2+bx+c = a\,(x-x_1)\cdot(x-x_2)$

dove $x_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

Quindi

$$\frac{px+q}{ax^2+bx+c} = \frac{1}{a}\left[\frac{px+q}{(x-x_1)\cdot(x-x_2)}\right]$$

Cerco due costanti $A, B \in \mathbb{R}$ tali che

$$\boxed{\frac{px+q}{(x-x_1)\cdot(x-x_2)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}}$$

Si prova che tali costanti A e B possono sempre essere trovati.

$$\implies \int \frac{px+q}{ax^2+bx+c}\,dx = \frac{1}{a}\left[A\int \frac{1}{x-x_1}\,dx + B\int\frac{1}{x-x_2}\,dx\right]$$

$$= \frac{1}{a}\left[A \log|x-x_1| + B\log|x-x_2|\right] + c$$

Caso $\boxed{\Delta = 0}$ $\quad ax^2+bx+c = a\,(x-x_1)^2$

con $\boxed{x_1 = -\dfrac{b}{2a}}$

$$\Longrightarrow \quad \frac{px+q}{ax^2+bx+c} = \frac{1}{a}\left[\frac{px+q}{(x-x_1)^2}\right]$$

Da qui vado alla ricerca di due costanti $A, B \in \mathbb{R}$ tali che

$$\boxed{\frac{px+q}{(x-x_1)^2} = \frac{A}{x-x_1} + \frac{B}{(x-x_1)^2}}$$

si prova che tali costanti A e B esistono sempre!

Una volta trovate le costanti, si ottiene

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \frac{1}{a}\left[A\int\frac{1}{(x-x_1)}\,dx + B\int\frac{1}{(x-x_1)^2}\,dx\right]$$

$$= \frac{1}{a}\left[A\cdot\log|x-x_1| + B\left(-\frac{1}{x-x_1}\right)\right] + c$$

caso $\boxed{\Delta < 0}$ In tal caso il polinomio al denominatore ax^2+bx+c è irriducibile!

Prima cosa da fare: riscrivere il numeratore come derivata del denominatore:

$$\frac{px+q}{ax^2+bx+c} = p\,\frac{\left(x+\frac{q}{p}\right)}{ax^2+bx+c} = \frac{p}{2a}\cdot\frac{2ax+2a\frac{q}{p}}{ax^2+bx+c}$$

$$= \frac{p}{\quad}\;\left(2ax+b\right)-b + 2a\frac{q}{p} \quad =$$

$$= \frac{p}{2a} \frac{(2ax + b) - b + 2a\frac{p}{q}}{ax^2+bx+c} =$$

$$= \frac{p}{2a} \frac{2ax+b}{ax^2+bx+c} + \frac{p}{2a} \cdot \left(-b+2a\frac{p}{q}\right) \cdot \frac{1}{ax^2+bx+c}$$

$\Longrightarrow$

$$\int \frac{px+q}{ax^2+bx+c} dx = \frac{p}{2a} \int \frac{2ax+b}{ax^2+bx+c} dx + \frac{p}{2a} \left(-b+2a\frac{p}{q}\right) \int \frac{1}{ax^2+bx+c} dx$$

(1° sostituzione)

$$= \frac{p}{2a} \int \frac{1}{t} dt \Big]_{t=ax^2+bx+c} + \frac{p}{2a} \left(-b+2a\frac{p}{q}\right) \int \frac{1}{ax^2+bx+c} dx$$

$$= \frac{p}{2a} \log|ax^2+bx+c| + \frac{p}{2a} \left(-b+2a\frac{p}{q}\right) \int \frac{1}{ax^2+bx+c} dx$$

A parte risolvo $\displaystyle\int \frac{1}{ax^2+bx+c} dx$

Nota che

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) =$$

$$= a\left(\left[x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right] - \frac{b^2}{4a^2} + \frac{c}{a}\right) =$$

$$= a\left(\left[x+\frac{b}{2a}\right]^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right) =$$

$$= a\left(\left[x+\frac{b}{2a}\right]^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right) =$$

$$= a\left(\left[x+\frac{b}{2a}\right]^2 + \frac{4ac-b^2}{4a^2}\right) =$$

$$= a\left(\left[x+\frac{b}{2a}\right]^2 + \underbrace{\frac{-\Delta}{4a^2}}_{>0}\right) \longrightarrow H = \frac{-\Delta}{4a^2} > 0$$

Da cui:

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{\left(x+\frac{b}{2a}\right)^2 + H}\,dx =$$

$$= \frac{1}{a\cdot H}\int \frac{1}{\frac{\left(x+\frac{b}{2a}\right)^2}{H}+1}\,dx =$$

$$= \frac{1}{a\cdot H}\int \frac{1}{\left(\frac{x+\frac{b}{2a}}{\sqrt{H}}\right)^2+1}\,dx = (*)$$

Nota che $z = \dfrac{x+\frac{b}{2a}}{\sqrt{H}} = \dfrac{1}{\sqrt{H}}\cdot x + \dfrac{b}{2a\sqrt{H}}$

$$\Rightarrow z' = \frac{1}{\sqrt{H}}dx$$

$$\circledast = \frac{1}{a \cdot H} \cdot \sqrt{H} \int \frac{\frac{1}{\sqrt{H}}\, dx}{\left(\frac{1}{\sqrt{H}}x + \frac{b}{2a\sqrt{H}}\right)^2 + 1} \underset{\text{1° sost}}{=}$$

$$= \frac{1}{a\sqrt{H}} \cdot \left. \int \frac{dz}{z^2+1} \right]_{z = \frac{1}{\sqrt{H}}x + \frac{b}{2a\sqrt{H}}} =$$

$$= \frac{1}{a\sqrt{H}} \arctg\left(\frac{1}{\sqrt{H}}x + \frac{b}{2a\sqrt{H}}\right) + c$$

Esempio

$$\int \frac{2x+1}{3x^2+2x+1}\, dx = \circledast$$

$$\Delta = 2^2 - 4 \cdot 3 \cdot 1 < 0$$

riscrivo il numeratore come $(3x^2+2x+1)' = 6x+2$

$$\circledast = \int \frac{1}{3} \cdot \frac{6x+3}{3x^2+2x+1}\, dx = \frac{1}{3} \int \frac{6x+2}{3x^2+2x+1} + \frac{1}{3x^2+2x+1}$$

$$= \frac{1}{3} \int \frac{6x+2}{3x^2+2x+1}\, dx + \frac{1}{3} \int \frac{1}{3x^2+2x+1}\, dx$$

$$= \frac{1}{3}\int \frac{1}{t}\,dt\Big]_{t=3x^2+2x+1} + \frac{1}{3}\int \frac{1}{3x^2+2x+1}\,dx$$

$$= \frac{1}{3}\log|3x^2+2x+1| + \frac{1}{3}\int \frac{1}{3x^2+2x+1}\,dx$$

A parte

$$\int \frac{1}{3x^2+2x+1}\,dx = \int \frac{1}{3\left(\underbrace{x^2+\frac{2}{3}x}+\frac{1}{3}\right)} =$$

$$= \frac{1}{3}\int \frac{1}{\left(x^2+\frac{2}{3}x+\frac{1}{3^2}\right)-\frac{1}{3^2}+\frac{1}{3}}\,dx =$$

$$= \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{3-1}{3^2}}\,dx =$$

$$= \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}}\,dx =$$

$$= \frac{1}{\cancel{3}\cdot\frac{2}{\cancel{9}\,3}}\int \frac{1}{\frac{\left(x+\frac{1}{3}\right)^2}{\frac{2}{9}}+1}\,dx =$$

$$= \frac{3}{2}\int \frac{1}{\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)^2+1}\,dx = (*)$$

$$\boxed{\text{Se } \quad z = \frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}} = \frac{3}{\sqrt{2}}x + \frac{1}{\sqrt{2}} \implies dz = \frac{3}{\sqrt{2}}\,dx}$$

$$(*) = \frac{\cancel{3}}{2}\cdot\frac{1}{\frac{\cancel{3}}{\sqrt{2}}}\cdot\int \frac{\frac{3}{\sqrt{2}}\,dx}{\left(\frac{3}{\sqrt{2}}x+\frac{1}{\sqrt{2}}\right)^2+1} \overset{1^\circ \text{ sost}}{=}$$

$$= \frac{1}{\sqrt{2}}\left[\int \frac{dz}{z^2+1}\right]_{z=\left(\frac{3}{\sqrt{2}}x+\frac{1}{\sqrt{2}}\right)} =$$

$$= \frac{1}{\sqrt{2}}\operatorname{arctg} z + c =$$

$$= \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{3}{\sqrt{2}}x+\frac{1}{\sqrt{2}}\right) + c$$

Conclusione:

$$\int \frac{2x+1}{3x^2+2x+1}\,dx = \frac{1}{3}\lg|3x^2+2x+1| +$$

$$+\frac{1}{3}\frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{3}{\sqrt{2}}x+\frac{1}{\sqrt{2}}\right)+C$$

Esempio

$$\int \frac{2x+1}{3x^2+2x-1}\,dx$$

$$\Delta = 2^2-4\cdot 3\cdot(-1) = 4\,(1+3) = 4^2$$

$$x_{1,2} = \frac{-2\pm 4}{6} = \begin{cases} -1 \\ \frac{1}{3} \end{cases}$$

$$\Rightarrow\ 3x^2+2x-1 = 3\left(x-\frac{1}{3}\right)(x+1)$$

Da cui:

$$\frac{2x+1}{3x^2+2x-1} = \frac{1}{3}\left(\frac{2x+1}{\left(x-\frac{1}{3}\right)(x+1)}\right)$$

Cerco costanti A e $B \in \mathbb{R}$ tali che

$$\frac{2x+1}{\left(x-\frac{1}{3}\right)(x+1)} = \frac{A}{\left(x-\frac{1}{3}\right)}+\frac{B}{(x+1)} =$$

$$= \frac{A\cdot(x+1)+B\left(x-\frac{1}{3}\right)}{} =$$

$$(x-\tfrac{1}{3})\,(x+1)$$

$$= \frac{(A+B)\,x + (A-\frac{B}{3})}{(x-\frac{1}{3})\,(x+1)}$$

$$\iff \quad 2x+1 = (A+B)x + (A-\tfrac{B}{3})$$

dal principio di identità tra polinomi

$$\begin{cases} A+B = 2 \\ A-\frac{B}{3} = 1 \end{cases} \iff \begin{cases} B = 2-A \\ A - \frac{(2-A)}{3} = 1 \end{cases}$$

da cui $\quad A - \frac{2}{3} + \frac{A}{3} = 1$

$$A\left(1+\tfrac{1}{3}\right) = 1+\tfrac{2}{3}$$

$$\frac{4}{\cancel{3}}A = \frac{5}{\cancel{3}} \iff A = \frac{5}{4}$$

$$B = 2-A = 2-\frac{5}{4} = \frac{3}{4}$$

Da cui

$$\int \frac{2x+1}{3x^2+2x-1}\,dx = \frac{1}{3}\left[\frac{5}{4}\int\frac{1}{x-\frac{1}{3}}\,dx + \frac{3}{4}\int\frac{1}{x+1}\,dx\right]$$

$$= \frac{5}{12} \lg|x-\tfrac{1}{3}| + \frac{1}{4} \lg|x+1| + c$$

Trattiamo un altro caso particolare di integrale fratto

$\forall n \in \mathbb{N}$ considero

$$\int \frac{px+q}{(ax^2+bx+c)^n} \, dx$$

Sia $\Delta = b^2 - 4ac$

caso $\boxed{\Delta > 0}$ $\Rightarrow$ $ax^2+bx+c = a(x-x_1)(x-x_2)$

$$\Rightarrow \frac{px+q}{(ax^2+bx+c)^n} = \frac{1}{a^n} \frac{px+q}{(x-x_1)^n (x-x_2)^n}$$

Si prova che esistono $2n$ costanti

$A_1, A_2 \ldots A_n$ e $B_1, B_2 \ldots B_n$

tali che

$$\frac{px+q}{(ax^2+bx+c)^n} = \frac{1}{a^n} \cdot \left[\sum_{i=1}^{n} \frac{A_i}{(x-x_1)^i} + \frac{B_i}{(x-x_2)^i} \right]$$

$$= \frac{1}{a^n} \cdot \left[\frac{A_1}{(x-x_1)} + \frac{A_2}{(x-x_1)^2} + \ldots + \frac{A_n}{(x-x_1)^n} + \frac{B_1}{(x-x_2)} + \frac{B_2}{(x-x_2)^2} + \ldots + \frac{B_n}{(x-x_2)^n} \right]$$

Da cui

$$\int \frac{px+q}{(ax^2+bx+c)^n} = \frac{1}{a^n}\left[A_1 \int \frac{1}{x-x_1} + A_2 \int \frac{1}{(x-x_1)^2} + \dots \right.$$

$$\dots + A_n \int \frac{1}{(x-x_1)^n} dx + B_1 \int \frac{1}{x-x_2} + B_2 \int \frac{1}{(x-x_2)^2} +$$

$$\left. \dots + B_n \int \frac{1}{(x-x_2)^n} dx \right]$$

dove sappiamo determinare ogni integrale della somma

Caso $\boxed{\Delta = 0}$ $\Rightarrow$ $ax^2+bx+c = a(x-x_1)^2$

$$\Rightarrow \frac{px+q}{(ax^2+bx+c)^n} = \frac{1}{a^n} \cdot \frac{px+q}{(x-x_1)^{2n}}$$

Si prova che, esistano $2n$ costanti $A_1, A_2, \dots, A_{2n} \in \mathbb{R}$ tali che

$$\frac{px+q}{(ax^2+bx+c)^n} = \frac{1}{a^n} \cdot \left[\sum_{i=1}^{2n} \frac{A_i}{(x-x_1)^i} \right]$$

e proseguo come nel caso precedente

caso $\boxed{\Delta<0}$

prima riscrivo il numeratore come $2ax+b$:

$$\frac{px+q}{(ax^2+bx+c)^n} = \frac{p}{2a}\,\frac{2ax+\frac{p}{2aq}}{(ax^2+bx+c)^n} =$$

$$= \frac{p}{2a}\,\frac{2ax+b}{(ax^2+bx+c)^n} + \frac{p}{2a}\left(-b+\frac{p}{2aq}\right)\frac{1}{(ax^2+bx+c)^n}$$

Da cui

$$\int \frac{px+q}{(ax^2+bx+c)^n}\,dx = \frac{p}{2a}\int\frac{2ax+b}{(ax^2+bx+c)^n}\,dx + \frac{p}{2a}\left(-b+\frac{p}{2aq}\right)\cdot\int\frac{1}{(ax^2+bx+c)^n}$$

$$\underset{1^\circ\text{sost}}{=} \frac{p}{2a}\int\frac{1}{t^n}\,dt\Big]_{t=ax^2+bx+c} + \frac{p}{2a}\left(-b+\frac{p}{2aq}\right)\int\frac{1}{(ax^2+bx+c)^n}\,dx$$

$$= \frac{p}{2a}\,\frac{t^{-n+1}}{-n+1}\Big]_{t=ax^2+bx+c} + \frac{p}{2a}\left(-b+\frac{p}{2aq}\right)\int\frac{1}{(ax^2+bx+c)^n}$$

A parte

$$\int\frac{1}{(ax^2+bx+c)^n} = \frac{1}{a^n}\int\frac{1}{\left(\left[x+\frac{b}{2a}\right]^2+\frac{-\Delta}{4a^2}\right)^n}\,dx$$

(se $H = \frac{-\Delta}{4a^2} > 0$)

$$1 \quad \int \quad 1 \quad dx$$

(se $H = -\frac{\Delta}{4a^2} > 0$)

$$= \frac{1}{a^n \cdot H^n} \int \frac{1}{\left[\frac{\left(x + \frac{b}{2a}\right)^2}{H} + 1\right]^n} dx$$

$$= \frac{1}{a^n \cdot H^n} \int \frac{1}{\left[\left(\frac{x}{\sqrt{H}} + \frac{b}{2a\sqrt{H}}\right)^2 + 1\right]^n} dx = \circledast$$

$$\boxed{z = \frac{x}{\sqrt{H}} + \frac{b}{2a\sqrt{H}} \Rightarrow dz = \frac{1}{\sqrt{H}} dx}$$

$$= \frac{1}{a^n \cdot H^n} \cdot \sqrt{H} \int \frac{\frac{1}{\sqrt{H}} dx}{\left[\left(\frac{x}{\sqrt{H}} + \frac{b}{2a\sqrt{H}}\right)^2 + 1\right]^n}$$

$$\overset{1^\circ \text{ sost}}{=} \frac{1}{a^n H^n} \sqrt{H} \left[\int \frac{dz}{(z^2+1)^n}\right]_{z = \frac{x}{\sqrt{H}} + \frac{b}{2a\sqrt{H}}}$$

dove per quest'ultimo integrale ho la formula per risolverlo!

Esercizio

Studiare il carattere della successione definita ~~per~~ ricorrenza

$$\begin{cases} a_1 = \frac{1}{2} \\ a_{n+1} = a_n^2 \end{cases}$$

Considero $f(x) = x^2 \longrightarrow f(a_n) = a_{n+1}$

$f'(x) = 2x$

ma $a_{n+1} = a_n^2 \geq 0 \quad \forall n \in \mathbb{N} \Rightarrow$

la successione a_n è formata da termini positivi

Quindi considero $f(x) : [0, +\infty[\longrightarrow \mathbb{R}$

ma $2x > 0 \quad \forall x > 0 \Rightarrow$ $f(x)$ cresce in $[0, +\infty[$

$a_1 = \frac{1}{2} \qquad a_2 = a_1^2 = \frac{1}{4}$

$\Rightarrow a_2 < a_1$

$\Rightarrow f(a_2) < f(a_1)$

$\Rightarrow a_3 < a_2 \Rightarrow a_4 < a_3 \ldots$

Ossia $\quad a_{n+1} \leq a_n$

$\Rightarrow \quad (a_n)_{n\in\mathbb{N}}$ è successione monotona decrescente

essendo $\quad a_n > 0 \quad \forall n \in \mathbb{N}$

$$\Rightarrow \quad \lim_{n\to+\infty} a_n = \inf \{ a_n : n\in\mathbb{N} \} \in \mathbb{R}$$

Supponiamo che $\quad l = \inf\{ a_n : n\in\mathbb{N}\} \geq 0$

Da un lato $\quad \lim_{n\to+\infty} a_{n+1} = l$

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} f(a_n) = f(l)$$

essendo $f(x)$ continua.

$$\Rightarrow \quad l = f(l) = l^2$$

ossia $\quad l \geq 0$ deve risolvere l'eqne

$$l^2 - l = 0 \quad \begin{cases} l = 0 \\ l = 1 \end{cases}$$

Nota che $\quad l \leq a_n \quad \forall n\in\mathbb{N}$, essendo $a_1 = 1$

Nota che $l \le a_n \quad \forall n \in \mathbb{N}$, essendo $a_1 = \frac{1}{2}$

$\Longrightarrow \quad l = 1$ lo devo scartare

Conclusione

$$\boxed{\lim_{n \to +\infty} a_n = 0}$$